Dirac-Connes Operator on Discrete Abelian Groups and Lattices

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Abstract

A kind of Dirac-Connes operator defined in the framework of Connes' NCG is introduced on discrete abelian groups; it satisfies a Junk-free condition, and bridges the NCG composed by Dimakis, Müller-Hoissen and Sitarz and the NCG of Connes. Then we apply this operator to d-dimensional lattices.

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I Introduction

In a series of works, A. Dimakis and F. Müller-Hoissen(D&M) developed the noncommutative geometry(NCG) on discrete sets, whose foundation is a universal differential calculus as well as its *reduction* on a discrete set [1][2][3][4]. When a discrete set is endowed with a group structure, D&M's NCG coincides with the NCG devised by A. Sitarz [5][6]. In this case, the differential calculus and reduction can be formulated using the left-invariant forms [7]. With the merits being simple and intuitive mathematically, this approach of NCG is essentially a cohomological description of *broken lines* on discrete sets; therefore, it provides a natural framework to describe physical systems on discrete sets, i.e. classical or quantum fields on lattices [8]. In fact, D&M deduced the correct Wilson action of gauge field on lattices within their formalism in [8]. [a] However, neither D&M nor Sitarz paid much attention to the "fermionic" contents on discrete sets which would give some insight into the famous puzzle of chiral fermions on lattices [10] from a NCG point of view, though D&M have discussed the "representations" of their geometry in [1][3][11].

On the other hand, A. Connes finished formulating the axioms for his NCG after works [12][13][14][15][16] being accumulated. This approach of NCG essentially describes the classical differential geometry using the tools of operator algebra and generalizes this description into the realm of noncommutative algebras; despite of the requirement of a tremendous mathematical preparation, it has an intimate relation with "fermionic" contents of nature, for its key concept, a generalized Dirac operator to which we will refer as Dirac-Connes opertor in this paper, determines the metric structure on a noncommutative space.

In this work, we compose a kind of Dirac-Connes operator for discrete abelian groups, thus we determine a Connes' geometry as a spinor representation of D&M-Sitarz' geometry on these groups. This operator can be regarded as a generalization of the so-called "naturally-defined" Dirac operator for lattice fermions in our previous work [17]. This article is organized as follows. We review briefly different versions of NCG in Section II. Then we explore our Dirac-Connes operator in Section III. The "naturally-defined" Dirac operator is introduced in Section IV. Some discussions are put in Section V. Being deserved to be mentioned, some other authors have also attempted to introduce new intuitions into solving the problem of geometric description of spinor on discrete systems from NCG point of view. J. Vaz generalized Clifford algebra to be

[a] D&M applied their geometry in the exploration of integrable systems also, on which we will not touch in this article [9].

non-diagonal in spacetime [18]. Balachandran *et al* studied a solution in discrete field theories based on the fuzzy sphere and its Cartesian products [19].

II Noncommutative Geometries

We first introduce the concept of universal graded differential algebra $\Omega(A) = \oplus_{k=0}^{\infty}\Omega^k(A)$ upon a unital associative algebra A with $\Omega^0(A) = A$. $\Omega(A)$ is a free algebra generated by symbols $\{a, da : a \in A\}$ subject to the relations $d\mathbf{1} = 0$, $d(a_0a_1) - da_0a_1 - a_0da_1 = 0$. The latter will enable us to express elements in $\Omega(A)$ as linear combinations of monomials of the form $a_0da_1...da_n$. A differential d is defined by relations $d(a_0da_1...da_n) = da_0da_1...da_n$, $d(da_0da_1...da_n) = 0$ which are equivalent to the requirement that $d^2 = 0$, and one can check that d satisfies a graded Leibniz law

$$d(\omega_p\omega') = (d\omega_p)\omega' + (-)^p\omega_p(d\omega'), \forall\omega_p \in \Omega^p(A), \forall\omega' \in \Omega(A)$$

II.1 NCG on discrete sets, Reduction of Differential Calculus

Now let A be the algebra of all complex functions on a discrete set S whose elements are labeled by a subset of integer and denote by latin characters $i, j, k, ...$ A has a *natural basis* $\{e^i; i \in S, e^i(j) = \delta^i_j\}$ such that any function f in A can be decomposed as $f = \sum_{i\in S} f(i)e^i$. The algebraic structure of A can be expressed as

$$e^ie^j = e^i\delta^{ij}, \forall i, j \in S$$

and the unit is $\mathbf{1} = \sum_{i\in S} e^i$. D&M proved that under this circumstance, the universal differential algebra or differential calculus $(\Omega(A), d)$ on S satisfies that

Lemma 1 *1)Let* $e^{ij} = e^ide^j, i \neq j$*, then* $\{e^{ij}, i \neq j\}$ *is a basis of* $\Omega^1(A)$*;*
2)$e^{ij}e^{kl} = e^{ij}e^{jl}\delta^{jk}$*;*
3)$e^{i_1\cdots i_r} := e^{i_1i_2}e^{i_2i_3}...e^{i_{r-1}i_r}, i_k \neq i_{k+1}, k = 1, 2, ..., r-1$ *form a basis for* $\Omega^{r-1}(A), r = 3, 4, ...$*;*
4)$e^{i_1\cdots i_p}e^{i_{p+1}\cdots i_r} = e^{i_1\cdots i_pi_{p+2}\cdots i_r}\delta^{i_pi_{p+1}}, p = 1, 2, ..., r-p = 1, 2, ...$*;*
5)$de^{i_1i_2\dots i_r} = \sum_{k=0}^{r+1}\sum_{j\neq i_k, j\neq i_{k+1}} (-)^k e^{i_1\cdots i_kji_{k+1}\cdots i_r}$*, for* $r = 1, 2, ...$*;*
6)The cohomology groups of $(\Omega(A), d)$ *is trivial.*

The geometric interpretation of $e^{i_1\cdots i_r}, r = 1, 2, ...$ is the algebraic dual of a (r-1)-step broken line $i_1...i_r$; therefore, Lemma 1 has a simple and natural geometric picture on discrete sets.

A reduction or a reduced differential algebra of $(\Omega(A), d)$ is defined to be $\Omega(A)/\mathcal{I}$ in which the two-sided ideal $\mathcal{I}$ is generated by a specific subset of $\{e^{ij}\}$; in another word, $\Omega(A)$ is reduced to a more meaningful differential calculus modulo relations generated by setting e^{ij} in this subset to be zero.

II.2 NCG on discrete groups, Reduction of Left-invariant 1-Forms

First we introduce some notations. We will use G instead of S for a discrete set equipped with a group structure, denote its elements by $g, h, ...$ and write the unit of G as e. For all $g \in G$, $\bar{g} := g^{-1}$. Let $G' = G\backslash\{e\}$, and $\sum_g{}' := \sum_{g\in G'}$. The right(left)-translations induced by group multiplications on A are defined to be $(R_g f)(h) = f(hg), (L_g f)(h) = f(gh)$. Formally we write $\partial_g f = R_g f - f$. As for abelian groups, $R_g = L_g =: T_g$.

Left-invariant 1-forms in $\Omega^1(A)$ are defined as

$$\chi^g = \sum_{h\in G} e^h de^{hg}, \forall g \in G' \tag{1}$$

as well as $\chi^e = -\sum_g{}' \chi^g$ for convenience, such that

$$df = \sum_g{}' \partial_g f \chi^g \tag{2}$$

Lemma 2 *(Sitarz) Using left-invariant 1-forms only without appealing to the definition in Eq.(1), one can show that all the requirements on a differential calculus, linearity, nilpotent, graded Leibnitz rule, are guaranteed sufficiently and necessarily, if that* $\chi^g f = (R_g f)\chi^g$, $d\chi^g + \{\chi^e, \chi^g\} + \sum_{h\neq g}' \chi^h \chi^{\bar{h}g} = 0$, *together with Eq.(2) hold.*

Hence, Eq.(2) can be written as $df = -[\chi^e, f]$.

A left-invariant reduction is generated by specifying a subset of G' denoted as G'' and setting $\chi^g = 0, g \in G'\backslash G''$. Let $\sum_g{}'' := \sum_{g\in G''}$ and still $\chi^e = -\sum_g{}'' \chi^g$, then Eq.(2) becomes $df = \sum_g{}'' \partial_g f \chi^g = -[\chi^e, f]$.

II.3 K-Cycles, Junk and Distance Formula

Connes defines a *K-cycle* to be a triple $(A, \mathcal{H}, D)$ consisting of a *-algebra A, a faithful unitary representation π of A on a Hilbert space $\mathcal{H}$ and an (unbounded) self-adjoint operator D(Dirac-Connes operator) on $\mathcal{H}$ with compact resolvent, such that $[D, \pi(a)]$ is bounded for all $a \in A$. Parallel to reductions on discrete sets, an extension of π to a representation of the universal differential algebra $\Omega(A)$ on $\mathcal{H}$ making use of Dirac-Connes operator D is required for the purpose to introduce a meaningful differential structure on A. First, extend π to be a *-representation of $\Omega(A)$ in $\mathcal{H}$ by defining that

$$\pi(a_0 da_1 ... da_n) := \pi(a_0)[D, \pi(a_1)]...[D, \pi(a_n)]$$

Note that, since we do not care the involution property of differential algebra in this paper, our definition here omits a "i^n" from the conventional one for simplicity. Second, to make π be a differential representation, we define *Junk ideal* $\mathcal{J}^n = ker(\pi|_{\Omega^n(A)})$ and

$$\Omega_D(A) = \oplus_{n=0}^{\infty} \Omega_D^n(A), \Omega_D^n(A) := \pi(\Omega^n(A))/\pi(d\mathcal{J}^{n-1})$$

Junk ideal will become nontrivial if $n \geq 2$, namely that one has to consider $\pi(d\mathcal{J}^1) = \{\sum_j [D, \pi(a_0^j)][D, \pi(a_1^j)] : a_0^j, a_1^j \in A, \sum_j \pi(a_0^j)[D, \pi(a_1^j)] = 0\}$ to define $\Omega_D^2(A)$ well.

Lemma 3 *(Sufficient Junk-free condition in second order)*
If $D^2 \in \pi(A)'$ where $\pi(A)'$ is the commutants of $\pi(A)$, then $\pi(d\mathcal{J}^1) = \emptyset$.

Proof:
The statement can be verified by two identities $[a, b][a, c] = \{a, b[a, c]\} - b\{a, [a, c]\}$, $\{a, [a, b]\} = [a^2, b]$.
□

On the other hand, one can induce a metric $d_D(,)$ on the state space $\mathcal{S}(A)$ of A by Connes' distance formula

$$d_D(\phi, \psi) = sup\{|\phi(a) - \psi(a)| : a \in A, \| [D, \pi(a)] \| \leq 1\}, \forall \phi, \psi \in \mathcal{S}(A)$$

III Dirac-Connes Operator on Discrete Abelian Groups

In this section, we focus on discrete abelian groups G and try to define K-cycles $(A, \mathcal{H}, D)$ on them. As a prescription, we suppose that the translations $T_g, \forall g \in G$ can be induced from A into $\mathcal{H}$, and be denoted as

$\widehat{T_g}$ which satisfies

$$\widehat{T_g}\pi(f) = \pi(T_g f)\widehat{T_g} \tag{3}$$

Note that Eq.(3) is obvious if $\mathcal{H}$ is a free module on A. Now we point out that each reduction of $\Omega(A)$ can be realized by a junk $\mathcal{J}^1$; in fact, that $e^{gh} = 0$ can be implemented by requiring that $\pi(e^{gh}) = \pi(e^g)[D, \pi(e^h)] = 0$, a constraint on D. Notice that $\pi(\chi^g) = \sum_{h\in G} \pi(e^h)[D, \pi(e^{hg})] =: \eta^g$, and implement a left-invariant reduction by setting $\eta^g = 0, g \in G'\backslash G''$. We will just consider finite left-invariant reductions i.e. $\sharp(G'') < \infty$.

One can check that

$$\eta^g \pi(f) = \pi(T_g f)\eta^g \tag{4}$$

Accordingly, D can be formally written as $D = {\sum_g}'' \eta^g$ and there is

$$\pi(df) = [D, \pi(f)] = \sum_g{}'' \pi(\partial_g f)\eta^g \tag{5}$$

Inspired by Eq.(3), we require that η^g has the factorized form $\eta^g = \Gamma^g \widehat{T_g}, g \in G''$(without a summation) in which $\Gamma^g \in \pi(A)'$, thus

$$D = \sum_g{}'' \Gamma^g \widehat{T_g}$$

and Eqs.(4)(5) hold. Moreover, we require that D satisfies the Junk-free condition in Lemma 3, i.e. $D^2 \in \pi(A)'$. To gain a solution, first we need partition G'' into three intersectionless subsets $G'' = \Sigma \coprod T \coprod \bar{T}$ where Σ contains all 2-order elements in G'' and T, $\bar{T}$ contain other high order elements respecting that if $g \in T$ then $\bar{g} \in \bar{T}$; accordingly,

$$D = \sum_{\sigma\in\Sigma} \Gamma^\sigma T_\sigma + \sum_{t\in T} (\Gamma^t T_t + \Gamma^{\bar{t}} T_{\bar{t}})$$

Proposition 1 *If there holds the Clifford algebra $Cl(\sharp(G''))$*

$$\{\Gamma^s, \Gamma^t\} = 0, \{\Gamma^{\bar{s}}, \Gamma^{\bar{t}}\} = 0, \{\Gamma^s, \Gamma^{\bar{t}}\} = \delta^{st} \tag{6}$$

$$\{\Gamma^s, \Gamma^\sigma\} = 0, \{\Gamma^{\bar{s}}, \Gamma^\sigma\} = 0, \{\Gamma^\sigma, \Gamma^{\sigma'}\} = 2\delta^{\sigma\sigma'} \tag{7}$$

for all $s, t \in T$, $\bar{s}, \bar{t} \in \bar{T}$ and $\sigma, \sigma' \in \Sigma$, then D is a Junk-free Dirac-Connes operator.

Proof:

A straightforward computation shows that $D^2 = (\sharp(\Sigma) + \sharp(T))\mathbf{1}$ if all Γ^g subject to the above Clifford relations.

$\square$

Note that more general solutions can be gained by varying those non-vanishing anti-commutation relations in Eqs.(6)(7) by a scalar factor.

IV Dirac-Connes Operator on Lattices

Let G be a d-dimensional lattice as a discrete group $\mathcal{Z}^d$ where $\mathcal{Z}$ is the integer-addition group, and the elements in $\mathcal{Z}^d$ are d-dimensional vectors whose components are integers. Define unit vectors to be $\hat{\mu}, \hat{\mu}(i) = \delta_{\mu i}, \mu, i = 1, 2, ..., d$ and $(T_\mu^{\pm} f)(x) = f(x \pm \hat{\mu})$. Consider a reduction $G'' = \{\pm\mu, \mu = 1, 2, ..., d\}$, hence $df = \sum_{\mu=1}^{d}(\partial_\mu^+ f\chi_+^\mu + \partial_\mu^- f\chi_-^\mu)$. According to the previous Section, Dirac-Connes operator on $\mathcal{Z}^d$ is

$$D = \sum_{\mu=1}^{d}(\Gamma_+^\mu T_\mu^+ + \Gamma_-^\mu T_\mu^-)$$

where $\Gamma_\pm^\mu$ satisfy $Cl(2d)$ relations

$$\{\Gamma_\pm^\mu, \Gamma_\pm^\nu\} = 0, \{\Gamma_\pm^\mu, \Gamma_\mp^\nu\} = \delta^{\mu\nu}, \mu, \nu = 1, 2, ..., d$$

In [17], we called this operator(with a modification which will be pointed out in the next section) “natural-defined” Dirac operator on a lattice and proved in $d = 2$ that $d_D(,)$ coincides with conventional Euclidean distance on this lattice.

V Discussions

First, we call the Junk-free condition in second order a geometric square-root condition; for being more specific, we write it as $D^2 = N\mathbf{1}$ where N is a normalization integer. While $\tilde{D}^2 = \triangle$ is called a physical square-root condition when a laplacian is well-defined. On lattices, they are connected by the relation

$$\tilde{D} = D - {\sum_g}'' \Gamma^g \mathbf{1}, \tilde{D} = {\sum_g}'' \Gamma^g \partial_g$$

Second, M. Göckeler and T. Schücker pointed out that in a restricted sense, conventional lattice Dirac operator is not compatible with Connes' geometry, due to the contradiction of first-order axiom [20]. Our work on this aspect is in proceeding.

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